

Mathematical Learning Experiences: Leveraging Elementary Pre-service Teachers' Existing  
Perspectives to Support New Understandings

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### Abstract

The purpose of this phenomenological research study was to examine the phenomenon of living through struggle while learning mathematics on the part of pre-service elementary teachers. We elicited lived-experience descriptions and interviews from 46 participants enrolled in an elementary teacher education program over the course of two years. We chose to focus on struggle, as one of the research-based teaching practices recommends that teachers of mathematics engage their students in “productive” struggle. We argue that opening up an understanding of the variant ways struggle manifests in learning informs the work of teacher educators; namely, it supports our understanding of how these lived-struggle moments may influence subsequent perspectives of mathematics and mathematics teaching and learning pre-service teachers may bring with them. We end by considering the ways in which teacher educators can leverage this understanding of lived-struggle to support continued growth, such as drawing on these experiences to engage pre-service teachers in a critical examination of mathematics teaching and learning.

*Keywords:* learning experiences, lived struggle, mathematics education, phenomenology, pre-service teachers, productive struggle

\*This study has been reviewed and approved by an institutional review board.

### **Introduction**

The goal and challenge of teacher education is to help pre-service teachers develop from students in classrooms to effective teachers of classrooms. Ball (1990a) notes that this work comes with challenges, particularly in the area of mathematics teacher education. She states, “As a mathematics teacher educator, my goal is to help my students learn to do something different from—and better than—what they experienced as pupils in mathematics classes” (Ball, 1990a, p. 10). This is not an easy or straightforward task. One challenge to this work is the influence of pre-service teachers’ past experiences in mathematics on their perspectives of mathematics and mathematics teaching and learning. As Schoenfeld (1988) posits, students develop particular beliefs about mathematics “as a result of their experiences with mathematics” (p.151); these beliefs may be at odds with the discourses about teaching and learning that exist in their teacher education programs (Gellert, 2000; Lutovac & Kaasila, 2011; Smith, 1996). Rather than reject or ignore these seemingly competing forces, we seek to elicit and open up pre-service teachers’ past learning experiences in order to better understand what they bring to their teacher education program. Thus, we take a complexivist view in our research, viewing “tensions as necessary and productive sites of insight—not matters to be flattened out, but potential triggers for richer understandings” (Davis & Sumara, 2006, p.18). As such, we are interested in investigating these tensions to understand how pre-service teachers’ experiences and resulting perspectives related to mathematics align with research-based practices that are emphasized in our teacher education program. In particular, we are interested in how preservice teachers’ experiences align with the practice of supporting productive struggle in learning mathematics. We recognize that, for many, the idea of struggle in any way may be perceived as something to be avoided in mathematics (Hiebert & Wearne, 2003). We hope to open up and investigate our pre-service teachers’ lived experiences with struggle in learning mathematics in order to better understand the felt nature of

them. In this way, we hope to respond to our pre-service teacher learners' prior experiences in ways that support their continued growth as teachers of mathematics.

### **Background Literature**

In *Adding it Up: Helping Children Learn Mathematics* (National Research Council, 2001), the National Research Council demonstrates the shifts that have occurred in with regards to what it means to be mathematically proficient. This reformation of mathematical proficiency includes multiple strands: conceptual understanding, computational proficiency, adaptive reasoning, strategic competence, and productive disposition. Productive disposition refers to “the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics” (NRC, 2001, p. 131). These strands are connected; each supports and is supported by the other. For example, students who develop competence in reasoning about mathematics and in using their understanding of mathematical connections to solve new problems will see mathematics as something that makes sense and view themselves as sense-makers.

Unfortunately, for many in the general adult population—including preservice elementary teachers—dispositions towards mathematics are often negative rather than productive (Phillip, 2007). Research suggests that teachers' negative dispositions toward mathematics may be influenced by their past experiences as learners of mathematics (Drake, 2006; Phillip, 2007). These negative experiences and related perspectives affect how preservice teachers respond to what they learn and experience in teacher education programs (Ball, 1990a, 1990b; Charalambos, Panaoura, & Philippou, 2009; Jong & Hodges, 2015; Kaasila & Lauriala, 2012; Lutovac & Kaasila, 2011). Ball (1990a) states that, “The educator's goal, therefore, is to intervene in the inevitable continuity of experience in ways that affect its future quality and

direction” (1990a, p. 12). Thus, the work of teacher education must consider not only what pre-service teachers need to learn to be effective teachers of mathematics, but also address the issue of how pre-service teachers’ past experiences may interfere with or support their learning. This requires explicit efforts to understand pre-service teachers’ past experiences and how they may influence their development as effective teachers of mathematics.

The National Council of Teachers of Mathematics (NCTM) has played a leading role in providing guidance for a vision of effective mathematics teaching and learning, i. e., for where pre-service teachers are headed. In 2014, NCTM published *Principles to Actions: Ensuring Mathematical Success for All* which describes a vision of what is necessary to support all students in learning mathematics. *Principles to Actions* outlines “a set of strongly recommended, research-informed actions for all teachers, coaches, and specialists in mathematics”(2014, p. 4). In doing so, NCTM draws on current research on mathematics teaching and learning to update and elaborate the guiding principles outlined in 2000 *Principles and Standards for School Mathematics*. One of these guiding principles is Teaching and Learning. NCTM states:

An excellent mathematics program requires effective teaching that engages students in meaningful learning through individual and collaborative experiences that promote their ability to make sense of mathematical ideas and reason mathematically. (p. 5)

Classroom interactions between teacher and student are key to students’ opportunities to make sense of and reason about mathematics (Ball & Forzani, 2011; Thames & Ball, 2013).

Thames and Ball argue that to improve teaching, mathematics teacher educators need to establish agreement “on some core practices that teachers should be able to perform” (2013, p. 38). Other researchers have made similar assertions that teacher education should focus on practice and some have articulated various sets of “high-leverage” or “core practices” that teachers should understand and develop (Ball & Forzani, 2011; Forzani, 2014; Grossman & McDonald, 2008;

McDonald, Kazemi, & Kavanaugh, 2013). While a practice-based perspective focuses on the work of effective mathematics teaching, there is still a recognition that teacher education must also “attend to the knowledge and orientations that underlie effective teaching” (Ball & Forzani, 2011, p. 19).

In *Principles to Actions*, NCTM puts forward a set of eight research-informed teaching practices that represent “a core set of high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics” (2014, p. 9). In 2017, the Association of Mathematics Teacher Educators (AMTE) released the *Standards for the Preparation of Teachers of Mathematics* developed, in part, to “articulate a national and comprehensive vision for the initial preparation of teachers of mathematics in Pre-K–12” (2017, p. x). This document also states the need for well-prepared teachers to understand and implement effective teaching practices “such as those described in *Principles to Actions*” (Association of Mathematics Teacher Educators, 2017, p. 15). One of the practices suggested by NCTM and AMTE is support productive struggle in learning mathematics. NCTM elaborates on this practice: “Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggles as they grapple with mathematical ideas and relationships” (2014, p. 10). In this paper, we focus our examination on how pre-service teachers’ lived experiences in mathematics and the perspectives that result from them align with the core teaching practice of supporting productive struggle in mathematics.

### **Productive Struggle**

The notion of struggle leading to learning is not new or exclusive to the study of mathematics. Dewey (1929, 1910/1997) and Poyla (1957) both argued that the effort of working through a problem can lead to new learning. Cognitive learning theorists (Hatano, 1988; Piaget,

1960; Skemp, 1971) suggest that struggle plays an important role in the learning process, such that new learning results from the resolution of disruptions to connections and understandings.

The equilibrium model (Piaget, 1977; Steffe, 1991; von Glasersfeld, 1991), for example, describes the ways in which learners act to dissipate perturbations in the environment. This is portrayed as containing a goal, activity, feedback, and results. In this process, a learner has the goal of dissipating a perturbation. Through cognitive activity and feedback, the perturbation is either alleviated (reaches equilibrium) or the cycle continues until equilibrium is reached.

VanLehn, Siler, and Murray (2003) examined the role of impasse—“when a student realizes that he or she lacks a complete understanding of a specific piece of knowledge” (2003, p. 220)—in learning and concluded that impasse is “strongly associated with learning” (2003, p. 244). They theorize that reaching an impasse motivates the student to seek to address his or her incomplete understanding.

Kapur puts forth the construct of productive failure, the notion that “engaging students in solving complex, ill-structured problems without the provision of support structures can be a productive exercise in failure” (Kapur, 2008, p. 379). Summarizing research in this area, he notes that results support “an argument for a delay of structure in learning and problem-solving situations” (Kapur, 2008, p. 381). Productive failure posits that, rather than providing external structure as learners engage in working on ill-structured problems, there is benefit from allowing learners to engage these problems without any support.

Research results support the benefits of learning through some form of struggle (Hiebert & Grouws, 2007; Kapur, 2010, 2011). For example, Kapur (2010, 2011) and Kapur and Bielaczyc (2012) conducted studies examining the effects of a productive failure instructional design in mathematical problem solving and found that, although students in the productive

failure groups initially struggled to reach a solution, they outperformed their peers on post-tests, demonstrated greater flexibility in solutions, and demonstrated greater transfer of knowledge. Kapur concluded, “there is efficacy in persistence itself even though it may not lead to success in performance” (Kapur, 2010, p. 545).

Summarizing research on the effects of classroom teaching on student learning, Hiebert and Grouws (2007) argue that a key feature in mathematics instruction that promotes mathematical proficiency, including conceptual understanding, is providing opportunities for students to “struggle with important mathematics” (p. 387), meaning “that students expend effort to make sense of mathematics, to figure something out that is not immediately apparent” (p. 387). The source of struggle is making sense of mathematical ideas and connections (rather than non-mathematical features, e.g., difficult wording) and must center on mathematics that is within the reach of the learner. Further, the learner must have prior knowledge that can serve as an entry point into the mathematics as a basis for making sense of new ideas (Hiebert et al., 1996; Hiebert & Grouws, 2007).

While results of research may support the importance of struggle in learning, it is difficult for teachers to engage students in productive struggle. Many teachers want to remove struggle from students, particularly in mathematics (Stein, Smith, Henningsen, & Silver, 2009; Stigler & Hiebert, 2004). To promote productive struggle, however, teachers should resist the urge to immediately step in and resolve students’ difficulties (Kapur, 2011). This runs counter to conventional wisdom, in which struggle in mathematics is often viewed negatively (Hiebert & Wearne, 2003). Such a perspective stems from a cultural view of mathematics teaching and learning that emphasizes following steps and procedures demonstrated by the teacher (Stigler & Hiebert, 2004). Mathematics that engages students in productive struggle requires a different



perspective of what it means to do, and consequently teach, mathematics (NCTM, 2014). Hiebert and Grouws (2007) explicitly distinguish this type of mathematical work from that which may be a familiar experience for many: “By struggling with important mathematics we mean the opposite of simply being presented with information to be memorized or being asked only to practice what has been demonstrated” (p. 388). Schoenfeld (1988) discusses the “difference between becoming competent at performing the symbolic manipulation procedures in a mathematical domain and grasping the underlying mathematical ideas in that domain” (p. 146). In the former view, mathematics is a static body of knowledge and the learner is a passive participant. The role of the teacher is to offer clear explanation of what to do and provide many examples of how to do it. Productive struggle reframes the discipline as dynamic and connected, positioning the learner as an active participant in forming and making sense of mathematical ideas (Schoenfeld, 1988; 1992).

### **Context/Motivation for the Study**

In this study, we chose to focus on productive struggle because it is a fundamental “feature of teaching that consistently facilitates students’ conceptual understanding” (Hiebert & Grouws, 2007, p. 387). Our goal as teacher educators is to support our pre-service teachers to understand and implement the core practice of supporting students in productive struggle in mathematics. However, we acknowledge that our pre-service teachers are not likely to have experienced this practice as learners. Thus, although we may attempt to promote this understanding in teacher education, we recognize that their prior experiences as mathematics learners may have not align with productive struggle, possibly influencing the ways in which they respond to the learning experiences we create. Ball notes if the learning from pre-service teachers’ past experiences does not fully support their development as teachers, teacher

educators must “interrupt, to break in, what is otherwise a smooth continuity from student to teacher” (Ball, 1990a, p. 11). Yet, this does not mean that we discount these past experiences because, as Ball continues, “It is that past experience, however reinterpreted, that necessarily provides the support and impetus for future learning” (Ball, 1990a, p. 12). Thus, we face a dilemma:

If the principle of continuity of experience is inevitable, what does that imply for the educator, one who wishes to shape and affect others’ futures? The responsibilities are twofold. First educators must judge what prior learnings can contribute to future growth and which may impede it. This implies a need to examine what learners bring—what they already know, believe, assume and are inclined to do. Educators must also have a vision of where learners are headed and what ideas, beliefs, attitudes, and dispositions are likely to prove useful for moving in that direction. (Ball, 1990a, p. 12).

As teacher educators, we noted variance in our prospective teachers’ responses to teacher education—they articulate different perspectives and demonstrate different reactions to new perspectives on teaching and learning mathematics. We wished to make sense of and respond to the variance, honoring what perspectives students currently hold and bring to teacher education and understand the experiences that were origins of those perspectives while using this to move forward. In our attempt to understand the perspectives our pre-service teachers brought with them to their teacher education program related to teaching and learning mathematics, we used an instrumental case study method (Stake, 2005). Case study research allows researchers to focus on the complexities of a phenomenon within a single case and across multiple cases (Baxter & Jack, 2008; Stake, 2005). Stake defines the purpose of “instrumental cases” as those that “provide insight into an issue or redraw a generalization. The case is of secondary interest, it plays a supportive role, and it facilitates our understanding of something else” (Stake, 2005, p. 445). In this initial study, summarized below, the “something else” was to understand what

perspectives pre-service teachers bring to their teacher education program and how these do or do not align with notions of productive struggle.

### **Initial Perspectives Students Bring to Teacher Education**

Generally, we noticed that our pre-service teachers' comments related to the nature of mathematics and what it means to do mathematics fell into two broad themes: *math as a recipe to follow/practice makes perfect* and *math is something that makes sense*. Those who perceived math as a recipe often asserted that they learn best if they can see math written out, step-by-step and follow this example. Several vocalized that they needed many examples and repeated practice in order to understand. Some of the phrases that we often heard included: "need lots of practice to understand," "learn by having examples presented to me," "learn by watching the process and doing examples on my own," and "need to see steps."

Math as a recipe reflects what Schoenfeld (1988) described as a view of mathematics as performing symbolic manipulation procedures and the act of teaching as presenting an explanation followed by multiple examples. Evident in these statements was a sense of learner passivity/lack of agency. These comments describe being *presented* with explanations and examples and *shown* steps. There is no evidence of the learner using their own knowledge to make sense of and solve tasks. Even students who explicitly expressed positive perceptions of themselves as learners of mathematics demonstrated lack of agency (e.g., "I enjoy math. It either comes quickly or not at all" or "can catch on quickly and understand it as long as it is explained well"). In many cases, the comments expressed an "I get it or don't" perspective; there was little evidence in the comments that there is something that could be done if they do not immediately get it.

Some of our pre-service teachers' comments reflected that they felt some agency in mathematics, that it was something that made sense and that they could make sense of it with effort. For example, one described herself as a slow learner, but "if I put the time and effort into it, I can figure it out." Others talk about feeling satisfaction from working to learn something new: "I like coming up with an answer after working hard to solve." Some students made comments indicating that they had recent experiences that resulted in some shift in their perspectives of themselves as mathematics learners. These comments reflected a perspective that mathematics is something that can be achieved with effort; that they have some role in making sense of mathematics.

### **Perspectives That Align with Productive Struggle**

Realizing the variant perspectives our pre-service teachers brought with them to teacher education led us to consider more carefully where we wanted them to go as teacher educators. We began by considering the specific instructional skills/strategies they would need to learn in order to implement the practice of supporting students in productive struggle. We first examined the literature to determine instructional features/teaching strategies that are often identified as supporting the practice of engaging students in productive struggle (e.g., Doerr, 2006; Engle, 2006; Franke et al., 2015; Gresfali et al., 2009; J. Hiebert & Grouws, 2007; Kapur & Bielaczyc, 2012; NCTM, 2014; Stein et al., 2009; Warshauer, 2015b, 2015a). We categorized these into three broad areas: classroom culture/environment, treatment of mathematics, and facilitating mathematical discourse. With these in mind, we then articulated perspectives/beliefs that align with these features. Table 1 summarizes these strategies and related perspectives.

Table 1

*Instructional features/teacher strategies that support productive struggle and related perspectives.*

<b>Instructional Feature/Teacher Strategy</b>	<b>Related Perspectives</b>
<i>Classroom Culture/Environment</i>	
<ul style="list-style-type: none"> <li>Establishing a supportive learning environment (Kapur&amp;Bielaczyc,2012);</li> <li>Providing access to tools that support thinking process (NCTM, 2014)</li> <li>Allowing students time to struggle (NCTM, 2014; Warshauer, 2015a, 2015b)</li> <li>Explicitly encourage perseverance in sense-making (Gresfali et al., 2009; NCTM, 2014)</li> <li>Establishing shared authority in classroom for creating and validating the mathematics (Doerr, 2006; Franke, Turrou, Webb, Ing, Wong, Shin, &amp; Fernandez, 2016; Gresfali et al.,2009; NCTM, 2014; Warshauer, 2015a, 2015b)</li> <li>Communicating expectations that confusion and mistakes are natural and opportunities for learning (NCTM, 2014)</li> <li>Acknowledgement that struggle is an important part of learning mathematics (Warshauer, 2015a, 2015b)</li> </ul>	<p><i>All students can learn</i></p> <ul style="list-style-type: none"> <li>growth mindset/all students can improve understanding with effort</li> <li>the teacher is responsible for supporting the learning of all students</li> <li>a supporting, caring teacher is influential in creating a positive learning environment for students</li> </ul> <p><i>Nature of mathematics</i></p> <ul style="list-style-type: none"> <li>mathematics is more than just producing steps outlined by the teacher or the text</li> <li>authority lies in the mathematics; students can determine “correctness” by using the mathematics (not the teacher or text)</li> </ul> <p><i>Student agency</i></p> <ul style="list-style-type: none"> <li>learners are sense-makers in the mathematics/have agency</li> </ul> <p><i>Struggle is part of learning</i></p> <ul style="list-style-type: none"> <li>struggle is a beneficial feature of the learning process</li> <li>mistakes and errors are natural and can be opportunities for learning</li> </ul>
<i>Treatment of Content</i>	
<ul style="list-style-type: none"> <li>Implementation of tasks that promote reasoning and problem solving (NCTM, 2014; Stein, Smith, Henningsen, &amp; Silver, 2009; Warshauer, 2015a)</li> <li>Maintaining the cognitive demand of tasks during implementation (Stein, Smith, Henningsen, &amp; Silver, 2009)</li> <li>Anticipation of and plan for students’ solution approaches, including areas of struggle (NCTM, 2014; Stein, Smith, Henningsen, &amp; Silver, 2009)</li> <li>Opening up examination of mathematics to multiple solution approaches and ideas (NCTM, 2014; Stein, Smith, Henningsen, &amp; Silver, 2009)</li> </ul>	<p><i>Nature of mathematics</i></p> <ul style="list-style-type: none"> <li>mathematics is more than just producing steps outlined by the teacher or the text; mathematics is dynamic and connected</li> </ul> <p><i>Struggle is part of learning</i></p> <ul style="list-style-type: none"> <li>struggle is a beneficial feature of the learning process</li> <li>mistakes and errors are natural and can be opportunities for learning</li> </ul> <p><i>Student agency</i></p> <ul style="list-style-type: none"> <li>learners are sense-makers in the mathematics/have agency</li> </ul>
<i>Facilitating Mathematical Discourse</i>	
<ul style="list-style-type: none"> <li>Posing questions to scaffold and support but not rescue (Engle,2006; Franke, Turrou, Webb, Ing, Wong, Shin, &amp; Fernandez,2016; Hiebert &amp; Grouws, 2007; NCTM, 2014 Warshauer, 2015b)</li> <li>Listening to students’ ideas and pose questions to scaffold, support, elicit and build on their thinking</li> </ul>	<p><i>Struggle is part of learning</i></p> <ul style="list-style-type: none"> <li>struggle is a beneficial feature of the learning process</li> <li>mistakes and errors are natural and discussing them creates opportunities for learning</li> </ul>

Instructional Feature/Teacher Strategy	Related Perspectives
(Doerr, 2006; Engle, 2006; NCTM, 2014; Warshawer, 2015a, 2015b) <ul style="list-style-type: none"> <li>Facilitating discourse focused on explanation and justification (NCTM, 2014)</li> </ul>	<i>Student agency</i> <ul style="list-style-type: none"> <li>learners are sense-makers in the mathematics/have agency</li> </ul> <i>Nature of mathematics</i> <ul style="list-style-type: none"> <li>explanations require mathematical justifications (not just “what” but also “why”)</li> </ul>

For example, Kapur’s (2010, 2011) and Kapur’s and Bielaczyc’s (2012) work emphasized the importance of creating a learning environment that values persistence. Teachers can do so by explicitly emphasizing expectations of students that focus on effort and flexibility in solution methods and strategies rather than successful outcomes. NCTM (2014) and Gresfali and colleagues (2009) also argue for the importance of emphasizing perseverance in making sense of mathematics. Warshawer’s (2015a, 2015b) and Stein et. al’s (2009) work note the importance of teachers’ providing time for students to struggle with mathematics and making explicit the role that struggle plays in learning. Other research (e. g., Doerr, 2006; Franke et al., 2015; Gresfali et al., 2009) highlights the importance of making use of students’ contributions to the work of the mathematical community, valuing students’ thinking, and positioning their ideas as valid mathematical considerations. We categorized these and similar findings as instructional features related to classroom culture/environment. Perspectives aligned with instructional features related to the classroom culture/environment fall under four categories: all students can learn/growth mindset (teacher is responsible for supporting the learning of all students); student agency (learners are sense-makers of the mathematics); nature of mathematics (authority lies in the mathematics); and struggle is part of learning (mistakes are natural occurrences in the learning process).

Another category of instructional features/teacher strategies that support productive struggle relates to the treatment of mathematics content. One features in this category is the

selection of rich tasks that engage students and highlight the mathematics content to be learned (Kapur, 2008; National Council of Teachers of Mathematics, 2014; Stein et al., 2009; Warshauer, 2015a). The design of such tasks must be informed by clear understanding of students' prior knowledge, specifically that which relates to the current learning goals (Kapur, 2011). Further, implementation of the tasks must maintain the high cognitive demand of the task, as written (Stein et al., 2009). To do so, teachers must "resist the impulse to provide assistance or help" as soon as students experienced struggle (Kapur, 2011, p. 575). In addition, teachers should encourage and honor multiple approaches and ideas to solutions, encouraging open discussion of the mathematical merits of each (Kapur & Bielaczyc, 2012; National Council of Teachers of Mathematics, 2014; Stein et al., 2009). To facilitate this, teachers must anticipate the multiple ways students may approach a solution and plan for areas of struggle, considering how to support students without rescuing them and how to create opportunities to learn from students' developing understandings (National Council of Teachers of Mathematics, 2014; Stein et al., 2009; Warshauer, 2015a). Perspectives aligned with the instructional features related to treatment of mathematics content include: perspectives related to the nature of mathematics (mathematics is dynamic and connected); perspectives related to the role of struggle (struggle is a natural and important part of the learning process); and perspectives related to student agency in mathematics (students are mathematical sense-makers).

A third category, facilitating mathematical discourse, includes features such as posing questions and comments that scaffold and support students' efforts (Engle, 2006; Franke, Turrou, Webb, Ing, Wong, Shin, & Fernandez, 2016; Hiebert & Grouws, 2007; NCTM, 2014; Warshauer, 2015b); eliciting, listening and responding to, and making use of students' ideas (Doerr, 2006; Engle, 2006; NCTM, 2014; Stein et al., 2009; Warshauer, 2015a, 2015b); and facilitating

discourse focused on explanation and justification (NCTM, 2014; Stein et al., 2009). Teachers' orientations toward listening to students' ideas can influence students' agency and authority in mathematics classrooms (Doerr, 2006). Teachers who listen to understand rather than to evaluate and use this information to support students' mathematical growth promote a view of mathematics as something that is connected and logical and students as those who can make sense of ideas. Perspectives aligned with these features include: struggle is part of learning (examining errors creates learning opportunities); students have mathematical agency (learners are sense-makers); and nature of mathematical explanations (explanations require mathematical justifications).

Identifying instructional features and aligned perspectives enables us to consider productive and unproductive perspectives that pre-service teachers bring so that we may explore how they have developed and how we may engage in a critical examination of them. For example, the perspective indicated by many of our pre-service teachers that mathematics is a recipe to be followed represents the predominant cultural view of mathematics described above that mathematics is a set of rules and procedures demonstrated by the teacher and followed by the learner. Such a view does not closely align with a perspective of mathematics needed for embracing productive struggle, which views mathematics as something that makes sense, that students can create, and for which the authority lies in mathematical ideas and connections rather than the teacher or text. Very few of our pre-service teachers' initial comments reflected that students felt agency in mathematics, i.e., that they could make sense of it with effort.

We argue that experiences lead to productive and unproductive perspectives about teaching and learning mathematics. Further, pre-service teachers' experiences with mathematics influence the ways they respond to learning opportunities in teacher education and their



development as teachers of mathematics (Ball, 1990a; Charalambos et al., 2009; Jong & Hodges, 2015; Kaasila & Lauriala, 2012; Lutovac & Kaasila, 2011). As Ball argues, “Experiences may inhibit open-mindedness, freeze ways of looking, or engender undesirable attitudes. Experiences can therefore limit our possibilities for continued learning” (1990a, p. 11). We need to unpack past experiences in order to understand the origins of the perspectives our pre-service teachers bring and consider how those might influence their future paths. The following questions guided our subsequent phenomenological investigation, presented in this paper:

- What was (is) it for pre-service elementary teachers to live through struggle as mathematics learners?
- What patterns can be seen in our pre-service teachers’ stories of their mathematics experiences, both positive and negative?

An examination of these questions offers an entry into understanding, responding to, and leveraging pre-service teachers’ existing perceptions of mathematics that result from their experiences as learners and future educators. We hope to use the understandings gained from opening up lived struggle in mathematics as a lens for making sense of and enacting research-based teaching and learning practices, in particular, supporting students’ engagement with productive struggle in mathematics.

### **Phenomenological Research Methods**

Phenomenology, as a research method, aligns well with our goal to understand the various ways in which struggle while learning mathematics was (is) lived by our pre-service teachers (van Manen, 1990). Phenomenology investigates intentionality, meanings that exist in complex relations between people, things, and contexts. Vagle (2014) emphasizes, “phenomenologists must understand that they are not studying the subjective intentions, as in

purposes or objectives, of individuals but the ways in which meanings “come-to-be” in relations” (p. 112).

Specifically, this study used both interpretive (van Manen, 1990, 2014) and post-intentional (Vagle, 2010, 2014) forms of phenomenological research, which supports investigations regarding questions about living with, in, and through phenomena. Although there are several forms of phenomenology, both philosophically (e.g., Husserl’s transcendentalism, Heidegger and Merleau-Ponty’s existentialism, Ihde’s postmodernism) and methodologically (descriptive, interpretive, post-intentional), more recent interpretations problematize and reconceptualize techniques and assumptions that have proven problematic in qualitative inquiry (e.g., essencing, bracketing, describing). These methodologists emphasize the ontological aspect of experiencing phenomena (living through, being in) rather than the epistemological focus of descriptive and transcendental forms. Additionally, embodiment and variation is foregrounded rather than cognition and invariance (Ihde, 1993; Merleau-Ponty, 2002). There is not a prescribed method for conducting phenomenological research. However, we drew on Vagle’s (2014) five component process, including analyzing data using a whole-part-whole method of analysis. The research question driving our phenomenological investigation asked: *What was (is) it for pre-service elementary teachers to live through struggle as mathematics learners?* We also wondered: *What patterns can be seen in our pre-service teachers’ stories of their mathematics experiences, both positive and negative?*

## **Participants**

Forty-six elementary education majors participated in this study and were enrolled in courses taught by the researchers. Thirty were enrolled in a master’s level mathematics methods course during 2014 or 2015. The remaining 16 were enrolled in an undergraduate mathematics

content course for teachers during 2015. Those enrolled in the mathematics content course were in the first year of their program and did not have prior experiences with pedagogy courses or school-based experiences. Those enrolled in the methods course were towards the end of their program and completed or were simultaneously taking additional methods courses in literacy education, science education, and social studies education.

### Data Sources

Pre-service teachers' experiences were elicited through autobiographical (lived-experience) writing, classroom discussions, and interviews. It is common to use reflective writing prompts to better understand lived-experiences in phenomenological research (Dahlberg, Dahlberg, & Nyström, 2008; Vagle, 2014; van Manen, 2014). All 46 participants wrote about their mathematical learning experiences by responding to the prompts detailed in Table 2 from 2014 - 2015. Pre-service teachers (16) in the undergraduate mathematics content course were asked to respond to prompts asking them to describe themselves as mathematics learners and prospective mathematics teachers (see part a in Table 2). Pre-service teachers (30) in the master's mathematics methods course were asked to write a mathematics autobiography, describing their past mathematical experiences and pivotal moments (see part b in Table 2).

Table 2

#### *Course Assignment Prompts*

Course	Assignment Prompt
a. Mathematics Content for Elementary Teachers	Perspectives of Self as Learner and Teacher of Mathematics <ul style="list-style-type: none"> <li>Describe how you see yourself as a mathematics learner.</li> <li>Describe how you want to be as a teacher of mathematics.</li> </ul>
b. Elementary Mathematics Methods	Mathematics Autobiography <ul style="list-style-type: none"> <li>Write a mathematics autobiography focusing on key moments you have experienced as a student or as a teacher.</li> <li>Elaborate on what makes these moments stand out for you.</li> <li>Why do they resonate or continue to elicit feelings of joy or discomfort?</li> </ul>

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- What role did those around you (parents, teachers, siblings, classmates, etc.) play in helping shape your dispositions towards mathematics?
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During the spring of 2016, eight participants agreed to be interviewed by the researchers, providing more detail about their mathematical learning experiences. The interviews were semi-structured, starting off by asking the interviewee to “walk me through the [pivotal experience described in reflective writing] from your perspective. What stands out in your memory? Describe specific events that come to mind.” Interviewees were additionally probed to articulate the ways these moments felt for them.

Additional data sources included the researchers’ reflexive journal entries and analytic memos. Both researchers shared entries from their journals at weekly collaborative meetings which included reflections related to ongoing readings as well as our emerging understandings and personal reflections on the phenomenon. Reflexive activity (Finlay, 2002; Heidegger, 2008; Macbeth, 2001; Vagle, 2014) is a central practice in phenomenological research. We were guided by Vagle’s (2014) recommendations to write through moments of “connection” and “disconnection,” “assumptions of normality,” “bottom lines,” and moments when we found ourselves “shocked” (p 132-133). Analytic memos also served as a source of data. These are described as “a brief or extended narrative that documents the researcher’s reflections and thinking processes about the data” (Glesne, 2011; Miles, Huberman, & Saldaña, 2014, p. 95; Saldaña, 2013). These were crafted to capture decisions and thoughts related to emerging codes, operational definitions, and patterns that seemed to manifest. All data sources were transcribed and uploaded into Transana, a multi-user qualitative data analysis program that allowed the researchers to collaboratively organize and code the data.

## **Analysis**

Most phenomenological researchers recommend using a whole-part-whole method of analysis in phenomenological research (Dahlberg et al., 2008; Vagle, 2014; van Manen, 2014).

As Vagle (2014) emphasizes, manifestations of a phenomenon occur in the relationship between the parts and the whole of the research:

...whole-part-whole analysis methods stem from the idea that we must always think about focal meanings (e.g., moments) in relation to the whole (e.g., broader context) from which they are situated—and once we begin to remove parts from one context and put them in dialogue with other parts, we end up creating new analytic wholes that have particular meanings in relation to the phenomenon. (p. 97)

In our whole-part-whole reading, we began by reading all the data as a whole collection, withholding analysis. This was followed by multiple line by line readings for each participant. For the first line by line reading, we highlighted episodes of participants' stories and applied "tentative labels" – this is described by Saldaña (2013) as a holistic method of chunking a "large unit of data in the corpus to capture a sense of the overall contents and possible categories that may develop" (p. 141). After this first line by line reading, we realized that the mathematics learning experiences of the participants in our study fell into several broad categories: negative experiences, positive experiences, reflections on teaching, general reflections, and shift moments.

The next line by line reading continued to use holistic codes and sub-codes to further analyze episodes (moments) of pre-service teachers' stories. Related to negative experiences, we identified 27 ways in which these were articulated (e.g., course failure, loss of confidence, lived struggle). For the purposes of this paper, we describe the continued analysis that focused on one of these negative experiences and the phenomenon of interest in this study, lived struggle. This was the most detailed aspect of participants' negative experiences learning mathematics. However, we continue a similar process for the others (e.g., positive experiences, shift

moments) simultaneously to retain a sense of the variations in mathematics learning experiences to later put these in juxtaposition with lived struggle.

The next line-by-line reading sought to better understand the various ways struggle was lived. We used multiple coding methods simultaneously: in vivo coding (using participants own words), emotion coding (e.g., aversion, anxiety, stress), process coding (e.g., continuous struggle), and values coding (Saldaña, 2013). At this point we continued reading the data, but this time read across participants, “looking for...tentative manifestations” of lived struggle (Vagle, 2014). Integrated in this last phase, we kept returning to the larger data collection to make sense of these moments of lived-struggle within the broader mathematics learning experiences articulated. We ended by crafting anecdotes from participants’ stories (van Manen, 2014) to convey the multi-faceted nature of lived-struggle. The following manifestations of the phenomenon emerged as the most salient and are elaborated below: 1) moments of unproductive struggle, 2) shortcomings as a learner, 3) perceptions of what it means to do mathematics, 4) living struggle with others, and 5) positive (productive) struggle moments. We ended our analysis by connecting these ways in which struggle was lived with theories concerning struggle in mathematics, in particular, productive struggle.

### **Manifestations of Learning Mathematics and Lived Struggle (Results and Discussion)**

In this section, we first share pre-service teachers’ lived positive and negative experiences with mathematics for the purpose of understanding their experiences learning mathematics broadly. Next, to understand how our pre-service teachers lived struggle in mathematics, we open up and examine manifestations related to that particular phenomenon.

#### **Negative and Positive Learning Experiences**

Pre-service teachers' lived experiences learning mathematics generated nine themes concerning negative experiences, outlined in Table 3 below. Within each of these expressed negative experiences, pre-service teachers make further distinctions which are shared as examples in the second column of Table 3. Two of the nine themes concerned characteristics related to teachers: an inability to relay information and being unresponsive to needs of learners. Additional themes related to pre-service teachers' lived negative learning experiences include: caretaker attitudes, fast pacing, struggle/difficulty understanding, loss of confidence, an attitude of giving up, rote learning/single problem solving process, and feeling humiliated and embarrassed.

Table 3

*Themes Related to Negative Experiences*

<b>Negative Experiences</b>	<b>Examples</b>
Teacher's Inability to Relay Information	<ul style="list-style-type: none"> <li>• "in depth explanations not provided"</li> <li>• "I could never figure it out the same way he presented it"</li> <li>• "not being taught the proper way to work out problems"</li> <li>• "not able to present the materials to use in a way that I, and many others, was able to gain anything from the course"</li> <li>• "a lot of seat work with little discussion"</li> <li>• "worked examples not provided"</li> </ul>
Unresponsive Teacher	<ul style="list-style-type: none"> <li>• "he knew I was struggling and never actually did anything to assist me"</li> <li>• "I never felt like the teachers cared particularly that I didn't understand just so long as I passed the class, and I usually did with a C"</li> <li>• "my high school teachers didn't seem concerned and became frustrated when I didn't understand and couldn't comprehend what I didn't understand"</li> <li>• "much of the work I had to learn on my own, and the teachers did not seem to care much about if we were learning and how we felt about the course"</li> <li>• "the professors did not seem happy to be there at all, they did not clarify concepts, and they basically did not seem to care if students were struggling"</li> <li>• "she gave me a sticker and said she was proud of me for coming to talk to her...she did not attempt to answer my question"</li> </ul>
Caretaker Attitude	<ul style="list-style-type: none"> <li>• "Since math was something I struggled with I clung to my mom's views and perceptions of the subject, and damned it from the start, even when I was young. Knowing that my mom wasn't good at the subject made me</li> </ul>

	<p>feel like it was okay to be bad at it, and I didn't need to push myself because nothing was going to change"</p> <ul style="list-style-type: none"> <li>• "my grandma didn't enjoy math like she enjoyed reading, so I didn't worry with it either"</li> <li>• "my father is really good at math and would always tell me 'math is your friend,' however, my mom and I think different"</li> </ul>
Fast Pace	<ul style="list-style-type: none"> <li>• "I would go to class the next day and ask questions, trying to catch up, he would then move ahead to something new. I never felt that I had the concept before we proceeded along to something else. With this frustration I kept good grades but never felt my full potential"</li> </ul>
Struggle, Difficulty Understanding	<ul style="list-style-type: none"> <li>• "I studied hours upon hours the material and it never stuck"</li> <li>• "even when I do 'get' something I feel as though it's hard to keep a hold of as I am grasping a piece of paper in a hurricane that will blow away at any second"</li> <li>• "working the problems out myself seems to never work out the way I want them too and takes way too much time"</li> <li>• Math doesn't come easily for me: "I never seemed to be able to grasp mathematical concepts quickly, which was the most frustrating aspect of math for me because other subject came easy to understand," "I have to really take my time and focus on each step when solving a math problem"</li> </ul>
Loss of Confidence	<ul style="list-style-type: none"> <li>• "I stopped asking questions to my other gifted friends because they always seemed to think what we were doing was easy"</li> <li>• "I specifically remember my middle school principal telling me that it didn't make sense that my English and reading scores were SO HIGH when my math was almost below average, and I was obviously not focusing enough during math...any confidence I had was shot"</li> </ul>
Attitude	<ul style="list-style-type: none"> <li>• "In the past I attribute my lack of math skills to my parents and my elementary through high school teachers because they were not teaching in ways that I was understanding. I now know that it is not their fault, but it was myself who did not want to understand. I shut myself off. Instead of looking for different ways to figure out a problem I stopped and gave up"</li> </ul>
Rote Learning, One correct Process for Solving Problems	<ul style="list-style-type: none"> <li>• "I had one specific teacher who was a stickler for algorithms and always wanted our solutions to reflect her ways of thinking. I was often frustrated when my "creative" answers were marked wrong simply because they did not resemble hers"</li> <li>• "[sarcastically] I found early on that formulas were the key to success in math. If you had the right formula and could recognize when to use it then you were golden. All you had to do was plug in the numbers (who cared why it worked right?) and voila you have the right answer"</li> <li>• "I remember my earlier years of learning math in school as, here is the rule, now use that to solve this kind of problem. There was not an explanation for what the rule represented or why it worked"</li> <li>• "I was reprimanded because I didn't do it like she had taught us and she made me erase it and explained to the students to not pay attention to what I had just done because it would confuse them. I felt defeated"</li> </ul>
Humiliation, Embarrassment	<ul style="list-style-type: none"> <li>• "having to go to the board in trigonometry class and work out homework problems in front of the whole class. While the teacher may have thought this was a beneficial and helpful experience for her students, I hated it because I often struggled with the homework problems and dreaded going to class every day to be humiliated in front of my peers. So, this was a</li> </ul>



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	negative math experience for me growing up and I would say has attributed to my negative disposition towards math”
•	“I have experienced many classes where I am the only student that does not understand what is going on. I have had teachers that have verbally made me feel inferior to other students, in front of the entire class”
•	“I recall a math teacher I had in the 7 <sup>th</sup> grade. She thought you should know how to do the problems after showing you one time on the board but I need a little more help than that. If you made a mistake she would correct you and call you out in front of the class”

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Pre-service teachers articulated various themes concerning lived positive experiences.

Table 4 below shares ten of these including: feelings of accomplishment, the perceived positive qualities of former teachers, an appreciation for their own productive struggle, a sense of being good at math, family support, a positive disposition towards mathematics, supportive math learning community, the activity of teaching others, a perceived nature of math with correct answers, and valuing diverse strategies. Within each of these expressed positive experiences, students make further distinctions which are shared as examples in the second column of Table 4.

Table 4

*Themes Related to Positive Experiences, Examples,*

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<b>Positive Experiences</b>	<b>Examples</b>
Feeling of Accomplishment	<ul style="list-style-type: none"> <li>• Solving difficult mathematics problems; overcoming challenge</li> <li>• Viewing concepts from different perspectives</li> <li>• Eager to engage</li> </ul>
Qualities of Teacher	<ul style="list-style-type: none"> <li>• Encouraging: “made me feel like I was capable”</li> <li>• Patient, dedicated, and supportive</li> <li>• Encouraged productive struggle and questioning disposition</li> <li>• Fun and enjoyable</li> <li>• Helped me create a positive disposition towards math; elicited positive feelings</li> <li>• Provided resources, gave positive feedback, provided extra assistance/time</li> <li>• “I was not told ‘because it’s the rule’”</li> <li>• Made math engaging</li> <li>• Explained in a way I understand</li> </ul>
	<ul style="list-style-type: none"> <li>• “methodical in teaching...clearly worked through steps to solve example problems...I knew that if I could follow the steps he taught I would be able to successfully find the solution to the problems presented”</li> </ul>

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Positive Experiences	Examples
	<ul style="list-style-type: none"> <li>• “clearly explained concepts”</li> <li>• “gave us note cards to refer to for the different equations”</li> <li>• “teacher gave us exactly what we needed to know for the test”</li> </ul>
Productive Struggle	<ul style="list-style-type: none"> <li>• “I loved how empowered I felt when I could persevere and figure out a problem or concept”</li> <li>• “I know not only what it is to struggle with math, but more importantly, I know about the importance of student-student and student-teacher relationships as well as the necessity of transparency with productive struggle”</li> </ul>
Being Good at Math	<ul style="list-style-type: none"> <li>• Connectedness of math concepts</li> <li>• “I consider myself a ‘math person’...mathematics has always come easy for me”</li> <li>• “I flew through the [timed] problems with ease...from then on, I felt that I was ‘good at math’”</li> <li>• “I quickly and accurately worked the problems...it made me feel very proud of my ‘math skills’”</li> <li>• “math has always just made sense in my brain...I guess this is sort of to be expected, as math always came easy to my mom and dad”</li> <li>• “eager to complete problems and find the solutions”</li> <li>• “I have been super successful...passing these classes with A’s shows that I can truly learn and understand math”</li> </ul>
Helpful Family	<ul style="list-style-type: none"> <li>• “She kept asking me why I would perform the steps that I did and when I told her that I was following ‘the rule,’ she would explain what the rule meant and why it can be used”</li> </ul>
Positive Disposition	<ul style="list-style-type: none"> <li>• “My parents both love math...I think their positivity helped me find the enjoyment of math again”</li> <li>• “wonderful friends, teachers, and professors who not only helped me enjoy math, but also encouraged me to see myself as someone who was ‘good at math’”</li> </ul>
Supportive Mathematics Learning Community	<ul style="list-style-type: none"> <li>• “intellectual support of my professors and peers”</li> <li>• “my classmates were probably the most influential throughout my schooling other than my teachers...if we needed help with problems, we would turn to each other for guidance”</li> </ul>
The Activity of Teaching Others	<ul style="list-style-type: none"> <li>• “fond memories of watching him become excited about understanding a ‘creative’ way to look at a problem when the traditional path did not make sense to him”</li> <li>• “When I make it more hands on and interesting she is able to focus and understand more. When it is just worksheets, she dreads being there”</li> <li>• “I would sit and read her math lessons and go online and read further in order to come up with strategies for her to understand...if I didn’t understand, neither would my student”</li> <li>• “I was able to break it down into east steps that all 4 boys were able to comprehend”</li> </ul>
Perceived Nature of Math with Correct Answers	<ul style="list-style-type: none"> <li>• “I feel math is something that never changes. It is a constant and always has a straight up answer. That is something that makes math make more sense to me. I hate it when there are multiple answers for one question – it is very confusing for me”</li> </ul>
Valuing Diverse Strategies	<ul style="list-style-type: none"> <li>• “just because I think of a different way to solve a problem does not mean that it is wrong”</li> <li>• “I really enjoyed the new strategies to help students understand, such as build a ten, double facts, and counting strategies. It helped me realize that there are entirely different ways I could have been taught the material”</li> </ul>

We believe both positive and negative experiences can be leveraged as opportunities for engaging learners in considering productive struggle as part of their teaching practice. For example, negative experiences related to mathematics as being rote and the learner's role being to mimic the "correct procedure" can be used to consider the result of an alternative perspective, aligned with productive struggle, in which mathematics is a web of connected ideas that can be developed and refined by the learner. Similarly, some experiences that are perceived positively by students, such as being successful at implementing steps and procedures, might offer opportunities to discuss the ways in which this does and does not align with instructional features supporting productive struggle, such as eliciting and making use of students' mathematical thinking.

While all experiences, both positive and negative, have the potential to be leveraged to support pre-service teachers' understanding of productive struggle and how they may engage their students in this high-leverage practice, lived-moments that explicitly describe struggle, manifested either positively and negatively, and struck us as particularly relevant. In the sections that follow, we discuss in more detail how pre-service teachers lived struggle in mathematics, both positively and negatively. As we seek to open up these experiences, we weave in considerations about how they may result in perspectives that align/don't align with notions of productive struggle. Excerpts written as anecdotes (van Manen, 2014) are shared throughout that capture the lived nature of their struggle, created from participants' writings and interviews.

### **Lived Struggle**

Although struggle may conjure a negative memory for many readers, the pre-service teachers in this study lived struggle in both positive and negative ways. Struggle appeared in themes from both positive and negative experiences. In negative experiences, struggle

manifested as an unproductive endeavor, shortcomings as a learner, as related to preservice teachers' perceptions of what it means to do mathematics, and living struggle with others. The majority of participants described these experiences as continuous, lived through most of their mathematics journey. Pre-service teachers did not often articulate experiences in which struggle manifested positively. Those that did described feelings of pride after having struggled through a particularly challenging mathematical experience. A few explicitly named their lived struggle experiences as "productive" struggle. Each of the manifestations of lived struggle are explicated below.

**Moments of unproductive struggle.** The majority of our pre-service teachers, who described lived struggle in their mathematics learning, experienced unproductive struggle: struggle that did not lead to greater understanding but, rather, led to feelings of frustration, a desire to give up, and developing strong negative dispositions towards mathematics. Gretchen, for example described her efforts to work out problems and to seek help as being unsuccessful:

Working the problems out myself seems to never work out the way I want them to and takes way too much time. Math only seems to continue to become a more complex problem of not understanding and continued frustration. A large part of why I feel I struggle is my lack of confidence in myself as a learner. I don't believe in myself or feel comfortable in the processes involved. The words that come to mind when I think of how to describe what math means to me would be nothing short than anxiety, stress, and frustration. These feelings would also be on the higher level of emotion than just a slight emotion. In the past year these feelings have gotten worse. I have taken both elementary math courses required for the major and struggled to its highest in both classes. I just barely passed the first class, managed to move forward, but was unsuccessful in completing the second course. I was being thrown into information that I had not seen since I was in elementary school. It was a little intimidating! I tried one on one meetings with my professor and even study sessions with my classmates as well as my roommate who does have a passion for math. I studied hours upon hours; the material never stuck. This was one of the hardest classes I have taken and my experience in it has resulted in, for lack of a better word, hatred toward math.

The result of these unproductive struggles is a lack of confidence and an extreme “hatred” for mathematics. Her experience is even corporal in nature as she describes being “thrown into information” when she was not prepared. She mentions that mathematics is a “process” that is uncomfortable for her. Her struggles feel futile; there is no understanding or success gained from them. Perspectives that align with productive struggle, on the other hand, see persistence as part of learning. These perspectives hold that effort is what leads to growth. Further, her description of mathematics as being a process and information seems to suggest a perspective of it as a static body of information and Gretchen as a passive participant who must catch the information she is being thrown by the instructor. This corporeal distance marks a disconnect between her as a learning and the object of mathematics.

**Shortcomings as a learner.** Gretchen’s story describes how her unproductive struggle led to frustration and lack of confidence. Many of our pre-service teachers described experiences with struggle in similar terms, where they take on an identity as being a non-math person. This manifested as labeling themselves as “wrong,” having a personal shortcoming, or even being destined to be bad at mathematics. Dana described this as feeling “less than:”

I can remember at every grade level my fear of “math time.” I did not want to participate, be called on to answer a question, or switch my homework with the other classmates. **I felt less than** in the mathematics department and I have continued to bring those feelings with me even as a college graduate student.

For Dana, these feelings are temporal in nature: “math time” was a specific moment in her day that was wrought with fear. She did not want to engage and reveal her shortcomings to others. Her story also indicates that she was made to feel inferior (“less than”) in her college mathematics learning, a feeling that persists with her today. Wendy is another student who articulates a story about being an outsider, describing herself as being an “other,” as one for whom math is magic:

My professor once said there are students out there who look at math as magic. I AM one of those people. Why do formula's work? Who knows? It just does. Magic, right? How did the magician pull a bird out of his hat when he put in a bunny rabbit? Magic. I remember doing some homework for one of her classes last semester and afterwards having to explain how I did it. I desperately wanted to simply write "A magician never tells her secrets. Now for my next trick...".

Wendy's likening mathematics to "magic" seems to suggest that mathematics exists in a world that is not accessible to her. Even when she completes a mathematical task, the idea of explaining how she did it causes her anxiety and seems like a foreign request. Her struggle manifests as a barrier to mathematics: it hides behind the cloak of a magician's secrets. Others, like Sam and Regina, also describe a similar struggle to access mathematics:

**Sam:** I'll be the first to admit that mathematics is not a strong skill of mine. I have never stressed over any other subject as much as I have math. It feels like there is a barrier between me and the math information that I am trying to learn.

**Regina:** Math is something that has never been easy for me. Math became something so foreign to me and so hard to understand that I felt like I was reading another language! It is not an exaggeration to say that I can look at a story that was written in the French language and be able to decipher more of it than I could an algebraic expression.

Sam explicitly describes "a barrier between" herself and mathematics. Noteworthy is her use of the phrase "math information" as if it is something that exists outside of her and she cannot reach or access it. Regina likens it to a foreign language, believing that she could decipher French more easily than mathematics. Here, students seem to experience struggle as something that removes them from the learning of mathematics; something that keeps them outside of this activity. In other words, their struggle makes them "other." They don't express any agency in being able to break down the "barriers" and participate in the mathematical community.

This notion of being outside the mathematical community was experienced by others as isolating. They felt as if they were "the only one" who struggled. Maya describes the feeling of being alone among her peers:

Math is difficult for me. I have always struggled to keep up with material as it is presented in class. It has never come easy for me and I have always been the student in class asking a million questions and seeking out extra help. I have experienced many classes where I am the only student that does not understand what is going on.

Maya, like others, describes unproductive efforts to “keep up with material as it is presented in class.” She talks of asking questions and seeking out extra help. Her experience is lived in isolation: “I am the only student that does not understand.” Her description of mathematics is of “material” that is “presented,” again, describing a perspective of the nature of mathematics as something that is static, existing outside learners, and that they must somehow be able to absorb and reproduce. The fact that she perceives her struggle as being singular—I am the only one—suggests that struggle is not something that she experienced as a natural part of the learning process; rather than being embraced as an opportunity for growth, it was felt to be a mark of failure or “wrongness.”

Like Maya, Celine experienced similar feelings of isolation, but for her, these existed within her family relationships:

You know I have to get at least a 'C'. And I was the only one in my family that ever got the grades that I got. Everybody else was just over achievers. I mean they are still to this day super highly intelligent people. And I'm just like shut up. Just shut up. I don't want to hear how you know. I don't understand it.

Celine describes being “the only one in her family” who did not do well in mathematics. While Celine felt alone in her family due to her struggle, others cite their family as the reason behind their struggles. For these students, it is as if being bad at mathematics is in their DNA or an inherited trait, a perspective that Sara articulates:

For me, math has always been a struggle. As long as I can remember, I have always made a C in all my courses with the exception of Geometry. My father is really good at math and would always tell me “math is your friend.” However, my mom and I think different. I remember being able to fly through the basics such as multiplication, addition, subtraction, and division. But once letters were thrown in there, my understanding began to dwindle.

While Sara does not explicitly state that she inherited her struggles with mathematics from her mother, she does discuss her own and her mother's feelings in contrast to those of her father who she describes as "really good at math." Sara mentions this as if it is an explanation, almost a justification, for her lack of understanding and struggle. Whitney, describes similar views, and explicitly acknowledges the influence of her mother's perspective of herself as a mathematics learner on her own perspectives:

Ever since I can remember I have struggled with math and I think it centers around my test anxieties and ADHD. It was easy to be distracted by all the numbers on the page and for them to run together when trying to separate all the problems. I have always struggled with reading comprehension, and this lead to word problems really triggering my ability to **get lost in the problems**. I would tend to focus on the parts that weren't important and struggled with separating all the key elements of the problem into separate parts that would come back together to form the math problem that needed to be solved. My mom was always someone who struggled in math as well and she definitely paved a road lined with acceptance of hating math. She would always say I am so bad at math, and I will never get good at it! Since it was something I struggled with, I clung to her views and perceptions of the subject, and damned it from the start, even when I was young. Knowing that my mom wasn't good at the subject made me feel like it was okay to be bad at it, and I didn't need to push myself because nothing was going to change.

Whitney talks about specific issues that contributed to her struggle—ADHD, test anxiety, and difficulties with reading comprehension. Whitney also, however, acknowledges that what she perceived as her mother's acceptance of her own struggle with mathematics gave her justification for her own struggles. The result was she felt she had no agency—there was no need to try to improve because she was destined to be bad just like her mother. For Whitney, learning mathematics was a lost cause—"nothing was going to change."

In all of these stories, pre-service teachers take on the identity of being someone who is bad at/can't do mathematics. This stance demonstrates what Dweck (2006) describes as a fixed mindset—the belief that intelligence and ability are fixed traits. For those who have this mindset, the idea of struggle as being a natural part of the learning process would seem counter-intuitive.



Rather, in this perspective, struggle would be an indicator of not having a particular kind of/degree of intelligence or ability. Fixed mindsets are far too common in mathematics (Dweck, 2008). On the other hand, a growth mindset—a belief that, with effort, learning and understanding can develop—aligns with notions of struggle as being an integral part of learning. If pre-service teachers hold a fixed mindset when it comes to their own learning, how will they handle struggle in their future students' learning?

**Perceptions of what it means to do mathematics.** Other pre-service teachers' descriptions of their struggle experiences seem to closely connect to their **perceptions of doing mathematics**: mathematics consists of rules and memorizing, carrying out steps, completing tasks quickly. Similar perspectives were evident in earlier stories (i.e., Gretchen's use of "processes" of math and "math information;" Sam's use of "math information"). These perceptions emerged from the way they lived unproductive struggle or labeled themselves as doers of mathematics. In the stories that follow, the pre-service teachers' perceptions of mathematics are foregrounded as a way struggle was felt. For example, Wendy described success in mathematics as applying the right formula at the right time; the struggle came when she had to solve problems that were somewhat different from those she had practiced:

I found early on that formulas were the key to success in math. If you had the right formula and could recognize when to use it then you were golden. All you had to do was plug in the numbers (who cared why it worked right?) and voila you have the right answer. I will never forget a trigonometry class I had when I was in high school about 11 years ago. I can still see the teacher blah, blah, blahing at the front of the class. Me not understanding anything and my math partner understanding even less. We sat there terrified and tortured wondering how we were ever going to make it through homework and even worse the tests. I would spend hours at a friend's house nightly after I got off work from my part time job pouring over the homework trying to make sense of it all. I was getting it by the skin of my teeth. I understood enough to get through it and that was it. The worst part was when it came to the tests and was elated to find many problems that were just like the homework, but of course there were always 2 or 3 at the end that were nothing like what we had done in homework or class but by being able to do these

we were somehow demonstrating a higher level of understanding (and usually worth the most credit).

Salient in Wendy's story is a suggestion that she recognized that there are shortcomings of just applying formulas without understanding; the comment "who cared why it worked, right?" was made sarcastically. She talks about struggling to try to make sense on her own because she was not given this opportunity in class. However, her struggle was not about understanding the mathematics; it was about completing the assigned tasks. The result was learning that enabled her to only replicate the exact problems she had practiced. When questions were posed that required transfer, she felt betrayed by her teacher in a way and was unable to do so. Her phrase, "being able to do these we were somehow demonstrating a higher level of understanding" indicates that she wasn't quite buying into this belief, or at least didn't feel she was given the opportunity through classroom learning or homework assignments.

For others, their struggle stemmed from their inability to do mathematics quickly or in their head. Sabrina and Bri both describe such experiences:

**Sabrina:** I could never seem to do mathematics in my head like the other children could and this frustrated me.

**Bri:** I know that most of my struggle and delay in math was due to my quickness in recalling multiplication tables.

This aligns with views of doing mathematics as arriving at an answer and not having to think, suggesting a fixed mindset—mathematics is an innate ability. These views are counter to the idea that learning meaningful mathematics should involve some struggle, some effort to make sense of new ideas and connections.

Others talk about difficulties with the memorization or the rote application of rules and procedures they perceived as being needed to be successful in mathematics. Celine described her frustrations:

I remember my earlier years of learning math in school as, here is the rule, now use that to solve this kind of problem. There was not an explanation for what the rule represented or why it worked. I tried very hard to remember each rule and when to use it, but the further I got into math, the harder that approach was. I was incredibly frustrated with math classes and took only the classes that were required for graduation.

Here, the struggle was with remembering the rule and when to apply it. She recognized the shortcomings of a memorization approach—the more difficult the mathematics became, the harder this approach became. She seems to express a desire for understanding why something works. Sara, too, differentiates between following a procedure and understanding in mathematics:

Overall, my experience with math has been kind of dismal. I struggle with grasping concepts. I have to really take my time and focus on each step when solving a math problem, no matter if it is algebra, geometry, measurement, etc. I can “do” math, but it takes me a while to grasp the concepts, which leads to my frustration and then I give up.

Sara seems to suggest that there is a difference between “doing” math and understanding. Here, understanding is linked with concepts suggesting that “doing” is linked with procedures. Her frustration and struggle results from a lack of understanding the why behind procedures. Penny, also, describes struggles with a view of mathematics as applying formulas or carrying out particular procedures: “I was often frustrated when my “creative” answers were marked wrong simply because they did not resemble hers.” Penny’s moments of frustration suggests that she did feel some agency and recognized that she could use her understanding to solve problems in ways that did not necessarily reflect the procedures presented by the teacher. However, the fact that these were not only not recognized but labeled “wrong” resulted in frustration.

Struggles resulting from memorization approaches, not knowing why, and from having mathematical thinking discounted could be leveraged to engage pre-service teachers’ understanding of strategies and perspectives that support productive struggle. Embedded in many of these stories are what Schoenfeld (1988) described as unproductive beliefs that result from

teaching based in traditional cultural views of mathematics (e.g., mathematics understanding is demonstrated by quickly solving problems, only mathematically gifted people can discover or create mathematics, and success in mathematics is achieved by solving problems following the steps presented by the teacher). However, the fact that pre-service teachers experienced struggle related to these perspectives could potentially be leveraged to engage in a critical reflection of the shortcomings of such perspectives and open their thinking to alternative views that align with productive struggle.

**Living struggle with others.** Struggle also manifested through relationships: with family, peers, and teachers. For some, their struggles with mathematics led to tensions in their families: fights around completing homework, disappointment over low grades, etc. In some cases, pre-service teachers responded by closing up, giving up, and no longer sharing their struggles with family members. Patricia describes a particularly intense experience:

Math was always very difficult for me, starting in elementary school. When I was younger, there would always be fights in my house in the evening while I tried to complete my homework. Most of those arguments revolved around my math assignments. I did not understand how to do the problems past the basic ones that were modeled in class. I will never forget one night sitting at the head of the table, well past nine o'clock on a school night trying to complete my homework. I was in the fourth grade, tears were streaming down my face and both of my parents were extremely frustrated. I tried to answer a question my mom was asking me and the next thing I know I got a big *whap* on the back of the head. She knocked my glasses off, scared the life out of me and I immediately gave up. From that moment on, I was completely shut off to the thought of mathematics, it intimidated me.

Here, Patricia's struggles permeated the entire family. Her frustrations extended to her parents, and poignantly her mother's smack to her head. These frustrations represent being "punished" for struggling with mathematics. For Patricia, this interaction carried over to her approach to mathematics—she became intimidated by it and shut herself off.

Others felt the struggle through their relationships with teachers. Phoebe described feeling that her teachers did not care if she learned:

From fourth grade on my experiences were pretty negative. I never felt like the teachers cared particularly that I didn't understand just so long as I passed the class, and I usually did with a C. In ninth grade I got my first F grade ever on a report card in Algebra. My high school teachers didn't seem concerned and became frustrated when I didn't understand and couldn't comprehend what I didn't understand.

Similar to Patricia, Phoebe felt that her struggle to understand frustrated others who she felt were in a position to support her. In these learning environments, struggle is not treated as a natural, welcome part of the learning process; rather it is something undesirable and, in some cases, deserves to be punished. These moments described by pre-service teachers concerning the negative relationships that manifested primarily because of their struggle with mathematics are in conflict with a supportive learning environment (Kapur & Bielaczyc, 2012). In contrast, NCTM (2014) articulates that environments, in which expectations are communicated wherein confusion and mistakes are natural and provide opportunities for learning, are those that support productive struggle.

**Positive (productive) struggle moments.** While not predominant, there were a few cases in which students described experiences of lived struggle in positive terms. These most often manifested as feelings of accomplishment and pride. Pre-service teachers talked about working through “tough” tasks, feeling “proud of myself” as a result. Most often, these moments were connected to experiences in which they finally passed a class, correctly solved problems, or did well on a test when they had failed to do these things in the past. In these cases, it is not really clear whether the source of these feelings of accomplishment was learning the mathematics or from achieving an outcome like a grade or answer. Some stories, such as Susan’s do suggest feelings of accomplishment from developing understanding:

As a student, myself, I am very comfortable with my mathematics ability. I feel like, with a bit of persistence and studying, I can learn any concept in math and apply it in different situations. And it can feel rewarding when you “get” and learn a new concept.

In these cases, the idea of feelings of accomplishment resulting from persistence or perseverance is important. This perspective does align with productive struggle and can be leveraged to support pre-service teachers’ understanding of the practice. Susan’s experience takes the idea a bit further: struggle is productive and leads to learning.

One final anecdote describes a struggle experience that aligns with many features and perspectives we have identified as explicating productive struggle. This experience is shared by George:

My all time favorite college class was actually complex variables. It was really hard. I struggled with it. I will be the first to admit that I really struggled with it. I think we all struggled with it. But it was just enough of a challenge that we were able to go along with it. I know I personally went home and watched every YouTube video that I could. And we had study groups and stuff where we would do homework. I felt like I had accomplished something in that class; I felt proud. I made me realize I can do something even though I struggle with it. You can still do it. You just have to dedicate yourself to it. And my instructor, she knew, she knew everything about me. She knew every problem I did. She would say, “Oh, the book says this, you did it this way, but this way also works. What do you prefer?” Because it was sometimes questions that I would ask. And she would read them and go, “Oh, these are the same. This one is the same way but it’s a different approach to it and I never thought of it that way.” And that was something that made me feel good because I was able to figure it out from a different perspective. So, yeah. I was happy that I was able to figure everything out. I still struggled with it. And I know it’s okay to struggle. And you probably should struggle to learn effectively. She didn’t just tell us the answer or go through it. She would let us struggle or give us hints. Whereas other teachers I have noticed would say, “Oh, you’re struggling. This is the way that you do it. This is how you do it.” But she didn’t, she didn’t do that. And, I don’t know, if that made a difference or not. I feel that a lot of teachers feel like they’re a bad teacher if their students struggle, maybe.

The experience George described highlights many of the features and perspectives outlined in Table 1. For example, George shared that his instructor did not immediately step in and rescue students; rather, she allowed them time to struggle, providing hints to support their

efforts. George described his experience as being challenging but just within reach. As a result, he was motivated to persevere and even collaborated with peers to make sense of the mathematics, something that Kapur and Bielaczyc (2012) highlighted as important for supporting students efforts. Further, George's instructor acknowledged and valued his solutions approaches, even if they were not the ones she had initially considered. Thus, George felt that his ideas were valued and that he had mathematical agency. George's lived experience could be leveraged to examine what it means to live struggle in mathematics productively. Reflecting on the instructor's interactions with George can provide a lens for helping pre-service teachers see what the practice of supporting productive struggle might look like.

### **Conclusion and Future Directions**

For most of our pre-service teachers, struggle in mathematics was experienced negatively. They lived it as an unproductive endeavor that resulted in strong negative dispositions towards mathematics and themselves as doers of mathematics. Some took on an identity as a "non-math" person: one who is inherently unable to do or is destined to be bad at mathematics. These experiences in some cases even manifested in relationships—with family, with teachers—in which struggle resulted in frustration and even punishment. In many experiences, struggle was related to perceptions of doing mathematics as memorizing formulas and procedures, producing steps outlined by the teacher, and arriving at answers quickly. These experiences do not readily align with instructional features and perspectives that support productive struggle in mathematics. Yet, as teacher educators, we need to find ways to support them in learning how to implement this practice. In *Standards for the Preparation of Teachers of Mathematics* (2017), AMTE describes "well-prepared beginning teachers of mathematics" as those who "regard *doing mathematics* as a sense-making activity that promotes perseverance,

problem posing, and problem solving” (2017, p. 9) and have productive dispositions towards mathematics. Further, well-prepared beginning teachers of mathematics create learning opportunities and use teaching practices that:

- “provide access, support, and challenge in learning rigorous mathematics” to all students
- “foster growth mindsets among students about learning mathematics”
- and “acknowledge mistakes as critical for learning and help students view mistakes as important in the learning process for engaging in mathematics.” (2017, p. 13)

In addition, well-prepared beginning teachers of mathematics understand that students must engage in cognitively demanding tasks in order to support the development of their mathematical thinking and reasoning (Stein et al., 2009).

From our own inquiry into our elementary pre-service teachers’ lived experiences learning mathematics, we find that their experiences for the most part do not align with these practices and understanding. Ball (1990a, 1990b) and Schoenfeld (1988) both point out the influence of experience on pre-service teachers’ development and state the need for eliciting, interrupting, and working with these experiences. We agree with Ball that we should not discount these past experiences, because as Ball (1990a) states, “it is that past experience, however reinterpreted, that necessarily provides the support and impetus for future learning” (p. 12). We argue that these moments of struggle can be leveraged to help them to open up and engage in a critical reflection of their experiences with struggle in contrast with struggle lived productively as defined by NCTM and AMTE.

A responsive disposition on the part of mathematics teacher educators is a viable way to discuss and reflect on past experiences with struggle in mathematics and the resulting perspectives, and in turn, support the work of engaging pre-service teachers in research-based mathematics teaching practices, such as productive struggle. Rather than attempt to reject or



ignore students' experiences and perspectives, we suggest that there are potential benefits in eliciting, investigating and making sense of them to better understand what our students bring to teacher education. For example, understanding that some pre-service teachers lived struggle as unproductive, resulting in frustration and even hatred of mathematics helps us as teacher educators understand why they might instinctively wish to remove struggle from their future students. Understanding that some pre-service teachers experienced struggle as an indicator or evidence of being innately bad at mathematics helps us understand why they might see struggle as something to hide or gloss over in instruction rather than something that can promote new learning.

In attempts to build on our understandings, we are investigating pre-service teachers' reflections on teaching and learning in light of their experiences and continued learning in their elementary teacher education program. Preliminary analysis of the data indicates that many of their reflections are reactions to their own struggles or struggles they anticipate on the part of their future students (even from those that didn't struggle). We believe these beginning reflections on teaching and learning can be leveraged to help them better conceptualize and practice what it means to actually support struggle in ways that are beneficial for mathematics learning. Although pre-service teachers don't directly discuss removing struggle for their future learners, many articulate the belief that math learning should be fun and struggling learners should be made to feel good. It is unclear what actions they may take to make learning fun and support learners to "feel good," but they talk often about providing extra time, after school support, eliciting parental involvement, relating mathematics to the real world, etc. We worry that these perspectives could cause these teachers to remove struggle, or at least focus their efforts in ill-informed ways. We are also starting to notice that for pre-service teachers that

experienced intense struggle, and still are, that they are filled with anxiety about teaching math. Interestingly, they seem to understand what practices they should be engaging with to support learning but don't feel like they can do it. We are still analyzing these reflections and hope to learn more about the connections between lived-struggle, the types of perspectives developed from these experiences in regards to teaching, and the role we might take in supporting pre-service teachers to foster productive struggle in their future practice.

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